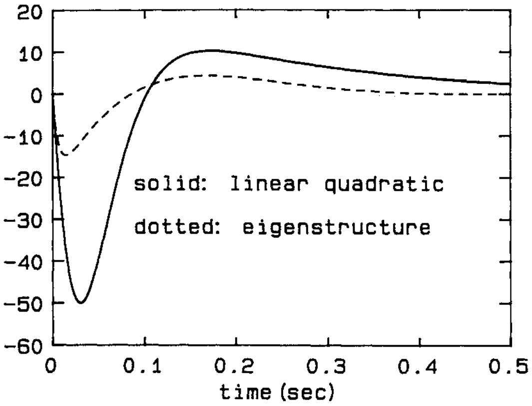

Fig. 2 Roll rate responses to 1-deg initial sideslip.

and integrated roll rate. The zero eigenvector entries are chosen so as to decouple the Dutch roll mode and the roll mode. The achievable eigenvectors are shown in Table 2 where we observe that we have obtained clearly defined Dutch roll and roll modes. The roll mode eigenvector is characterized by its largest entry corresponding to roll rate with very small entries corresponding to sideslip angle and yaw rate. The Dutch roll eigenvector is characterized by its largest entry corresponding to yaw rate with a very small entry corresponding to integrated roll rate and a small entry corresponding to roll rate. We would desire that the entry corresponding to roll rate be even smaller. Nevertheless, this new eigenstructure design should exhibit significantly improved decoupling between sideslip angle and roll rate. The sideslip and roll rate responses for the eigenstructure assignment autopilot to a 1-deg initial sideslip are shown in Figs. 1 and 2, respectively. We observe that a significant improvement is obtained in the decoupling between sideslip angle and roll rate. The peak value of roll rate is now − 14.2 deg/s, which compares with − 50.1 deg/s for the linear quadratic regulator design and represents an improvement of approximately 72%.

The maximum aileron deflections, aileron deflection rates, rudder deflections, and rudder deflection rates are 1.68 deg, 175.8 deg/s, 1.31 deg, and 119.6 deg/s, respectively, for the linear quadratic design and 1.90 deg, 211.0 deg/s, 3.16 deg, and 263.7 deg/s, respectively, for the eigenstructure assignment design. These maximum deflection rates are within the expected 400-deg/s limit for the advanced state-of-the-art electromechanical actuator described by Langehough and Simons.[3] The condition numbers of the modal matrices, which are a commonly used measure of eigenvalue sensitivity, are 200.77, 766.03, and 115.77 for the open-loop design, linear quadratic design, and eigenstructure assignment design, respectively. We observe that the linear quadratic regulator design exhibits a modal matrix condition number that is 3.8 times larger than the open-loop value, whereas the eigenstructure assignment design has a condition number that is almost half the open-loop value. Thus, the eigenstructure assignment design exhibits improved decoupling between sideslip and roll rate together with improved eigenvalue sensitivity, albeit at the expense of larger actuator deflection and deflection rates.

Acknowledgment

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Eigensystem Assignment with Output Feedback

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Introduction

EIGENVALUE and eigenvector assignment techniques have been extensively used in the active control design of linear, time-invariant systems. Numerous methods and algorithms involving both constant full-state feedback and output feedback have been developed.[1-14] Within the past decade, several iterative and noniterative algorithms[9-14] have been derived to exploit the freedom offered beyond the eigenvalue assignment by multi-inputs and multi-outputs, to either improve the performance of the closed-loop system or minimize the required control effort. Among the class of iterative methods is Kautsky's algorithm[9] that iteratively minimizes some robustness measure of the closed-loop system in terms of the conditioning of the closed-loop modal matrix through an orthogonal projection approach. Direct nonlinear programming techniques were used to minimize scalar robustness measures such as closed-loop conditioning[10,11] and closed-loop normality indices.[13] In the class of noniterative methods, a recent algorithm by Juang et al.[12] can be identified wherein closed-loop eigenvectors are chosen as such to maximize their orthogonal projection to the open-loop eigenvector matrix or its closest unitary matrix, thereby maximizing the robustness of the closed-loop system. A sequential algorithm by Maghami and Juang[14] can also be identified that utilizes Schur decomposition and Givens rotations to assign the desired closed-loop eigenvalues via full-state or output feedback. Many of the existing methods require full-state feedback that is not practical in lieu of the recent trends toward the erection and deployment of large flexible structures having thousands of degrees of freedom. Even those methods that can implement output feedback designs either do not take advantage of the full freedom of the system and/or are not computationally feasible.

In this paper, a new approach for the eigenvalue and eigenvector assignment of linear first-order, time-invariant systems is developed. The approach extends the procedure outlined in Ref. 12 by allowing the assignment of the maximum possible number of closed-loop eigenvalues via constant output feedback. The system is assumed to be fully controllable and observable, having full rank input and output influence matrices. The approach starts with the generation of a collection of bases for the space of attainable closed-loop eigenvectors corresponding to the desired closed-loop eigenvalues. The singular value decomposition (SVD) or QR decomposition are used

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to generate the collection of orthonormal bases. The coefficients corresponding to the bases are then computed through an algorithm based on subspace intersections, yielding an output feedback gain matrix. Furthermore, the freedom provided by multi-inputs and multi-outputs is characterized for further exploitation.

Formulation

The dynamics of a linear, first-order, time-invariant system may be represented as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where A is an $n \times n$ state matrix, B is an $n \times m$ control input influence matrix, $x(t)$ is an $n \times 1$ state vector, and $u(t)$ is an $m \times 1$ vector of the control inputs. The control algorithm is chosen to be a constant output feedback with p output measurements, i.e.,

$$u(t) = Gy(t) \tag{2}$$

in which G is an $m \times p$ gain matrix and $y(t)$ represents the $p \times 1$ output measurement vector defined as

$$y(t) = Cx(t) \tag{3}$$

where C denotes a $p \times n$ output influence matrix. The dynamics of the closed-loop system may then be written as follows:

$$\dot{x}(t) = [A + BGC]x(t) \tag{4}$$

The gain matrix G is to be designed such that a number of closed-loop eigenvalues are assigned to desired values. Davison[4] and Davison and Wang[5] have shown that for a fully controllable and observable system with an input matrix of rank $m(\leq n)$ and an output matrix of rank $p(\leq n)$, $\min(m+p-1, n)$ eigenvalues may be arbitrarily assigned via output feedback. Following the same assumptions, the present algorithm describes a systematic approach to the eigenvalue assignment problem with output feedback that takes advantage of the design freedoms.

Assuming that the number of output measurements p is larger than or equal to the number of inputs m, the right eigenvalue problem for the kth closed-loop eigenvalue is

$$[A + BGC]\psi_k = \mu_k \psi_k \tag{5}$$

where μ_k represents the kth desired closed-loop eigenvalue, and ψ_k denotes the corresponding closed-loop eigenvector. Rewriting Eq. (5) in a compact form gives

$$[A - \mu_k I_n \quad B]\begin{bmatrix} \psi_k \\ GC\psi_k \end{bmatrix} \equiv \Gamma_k \phi_k = 0 \tag{6}$$

in which I_n is an $n \times n$ identity matrix. Obviously, the nontrivial solution of the homogeneous equation, Eq. (6), is in the right null space of the matrix Γ_k. Expanding ϕ_k in terms of an orthonormal basis of the null space, one has

$$\phi_k = V_k c_k \tag{7}$$

where V_k represents a collection of orthonormal basis vectors of the right null space of Γ_k, and c_k is the vector of the corresponding coefficients. The orthonormal basis V_k may be obtained using the SVD or QR decomposition of matrix Γ_k. The details of these procedures have been outlined in Refs. 12 and 15.

Assume that s eigenvalues, where $s \leq \min(m+p-1, n)$, are to be assigned to the desired values that are either real or pairs of complex conjugates to guarantee the existence of a real gain matrix for the output feedback design. Furthermore, assume that the desired closed-loop eigenvalues do not coincide with any of the open-loop eigenvalues. Now, equations similar to Eq. (6) may be written for all the s closed-loop eigenvalues in a partitioned form, i.e.,

$$\Gamma_k \phi_k \equiv \Gamma_k \begin{bmatrix} \bar{\phi}_k \\ \hat{\phi}_k \end{bmatrix} \equiv \Gamma_k \begin{bmatrix} \bar{V}_k \\ \hat{V}_k \end{bmatrix} c_k = 0; \qquad k = 1,2,\ldots,s \tag{8}$$

where $\bar{\phi}_k$ and $\hat{\phi}_k$ are chosen from the null space V_k corresponding to ψ_k and $GC\psi_k$ of Eq. (6), respectively. It is noted that those vectors in the orthonormal bases V_k corresponding to a real closed-loop eigenvalue are real, and those corresponding to a pair of complex conjugate eigenvalues are complex conjugates, since both A and B are real. Furthermore, since B has rank of m and the system is fully controllable, the dimension of the null space of Γ_k is m, and subsequently each corresponding orthonormal bases V_k, $k = 1,2,\ldots,s$ has m vectors. Thus, each $\hat{V}_k$ is an $m \times m$ matrix. Comparison of Eqs. (6) and (8) requires that the gain matrix G satisfy

$$GC\bar{\phi}_k = \hat{\phi}_k; \qquad k = 1,2,\ldots,s \tag{9}$$

Substituting for $\bar{\phi}_k$ and $\hat{\phi}_k$ from Eq. (8) yields

$$GC\bar{V}_k c_k \equiv GW_k c_k = \hat{V}_k c_k; \qquad k = 1,2,\ldots,s \tag{10}$$

or in a compact form

$$\{GW_k - \hat{V}_k\} c_k = 0; \qquad k = 1,2,\ldots,s \tag{11}$$

Equation (11) indicates that a nontrivial solution for the coefficients c_k is possible if and only if the square matrix $\{GW_k - \hat{V}_k\}$ is rank deficient. The general form of matrix GW_k that would result in a rank deficient matrix $\{GW_k - \hat{V}_k\}$ may be written as

$$GW_k = (\Theta_k + \hat{V}_k); \qquad k = 1,2,\ldots,s \tag{12}$$

where Θ_k is an $m \times m$ rank deficient matrix to be determined later. Note that if the number of closed-loop eigenvalues to be assigned s is less than or equal to the number of outputs p, then the gain matrix can be computed from Eq. (11) for almost any arbitrary values of the coefficient vectors c_k, $k = 1,\ldots,s$. Consequently, in the later developments it is assumed that $s > p$. Furthermore, for simplicity of presentation, assume that the s closed-loop eigenvalues to be assigned are composed of r complex conjugate pairs, where $r = s/2$. Note that this assumption would not result in any loss of generality since real eigenvalues can be similarly treated as well. Rewriting Eq. (12) in a compact matrix form

$$G[W_1 \quad \ldots \quad W_s] \equiv GW = \left[(\Theta_1 + \hat{V}_1) \quad \ldots \quad (\Theta_s + \hat{V}_s)\right] \tag{13}$$

Here the partition matrices corresponding to complex conjugate eigenvalue pairs are positioned successively. A solution of the gain matrix G from Eq. (13) is attainable only if the rows of matrix $[(\Theta_1 + \hat{V}_1) \ \ldots \ (\Theta_s + \hat{V}_s)]$ are contained in the range space of the rows of matrix W. This can be achieved by requiring that the rows of matrix $[(\Theta_1 + \hat{V}_1) \ \ldots \ (\Theta_s + \hat{V}_s)]$ be orthogonal to the right null space of matrix W. Therefore, postmultiplying the right-hand side of Eq. (13) by matrix P, whose columns span the right null space of W, yields

$$\left[(\Theta_1 + \hat{V}_1) \quad \ldots \quad (\Theta_s + \hat{V}_s)\right] P = [\Theta_1 \quad \ldots \quad \Theta_s]\begin{bmatrix} P_1 \\ \vdots \\ P_s \end{bmatrix}$$

$$+ [\hat{V}_1 \quad \ldots \quad \hat{V}_s]\begin{bmatrix} P_1 \\ \vdots \\ P_s \end{bmatrix} \equiv \Theta P + Q = 0 \tag{14}$$

where P is an $m \times \bar{n}$ matrix, $\bar{n}$ is the dimension of the right null space of W, and Q is an $m \times (ms - p)$ matrix. In general, $\bar{n} = ms - p$. Note that an orthonormal set of basis vectors may be obtained using the SVD or QR decomposition. The eigenvalue assignment problem is now reduced to finding s rank deficient matrices $\Theta_1, \ldots, \Theta_s$ that satisfy Eq. (14). To guarantee a real-valued gain matrix, each pair of the coefficient matrices $\Theta_1, \ldots, \Theta_s$ corresponding to a pair of complex conjugate eigenvalues must be chosen to be complex conjugates. Moreover, to avoid complex arithmetics, Eq. (14) is expanded in terms of the real and imaginary components, yielding

$$\Theta_R P_R - \Theta_I P_I = -Q_R$$

$$\Theta_R P_I + \Theta_I P_R = -Q_I$$

leading to

$$[\Theta_{1R} \quad \Theta_{1I} \quad \ldots \quad \Theta_{rR} \quad \Theta_{rI}] \begin{bmatrix} \bar{P}_1 \\ \vdots \\ \bar{P}_s \end{bmatrix} \equiv \bar{\Theta}\bar{P} = \bar{Q} \tag{15}$$

in which

$$\bar{P} = \begin{bmatrix} P_{1R} + P_{2R} & P_{1I} + P_{2I} \\ -P_{1I} + P_{2I} & P_{1R} - P_{2R} \\ \vdots & \vdots \\ P_{(s-1)R} + P_{sR} & P_{(s-1)I} + P_{sI} \\ -P_{(s-1)I} + P_{sI} & P_{(s-1)R} - P_{sR} \end{bmatrix}$$

and

$$\bar{Q} = -[Q_r \quad Q_I]$$

The subscripts R and I denote the real and imaginary parts, respectively. Obviously, $\bar{P}$ is an $ms \times 2\bar{n}$ matrix and $\bar{Q}$ is an $m \times 2\bar{n}$ matrix. The coefficient matrices $\Theta_{1R}, \Theta_{1I}, \ldots, \Theta_{rR}, \Theta_{rI}$ may now be computed using an algorithm based on subspace intersections. This algorithm has two distinct, but similar, procedures depending on the number of pairs of closed-loop eigenvalues to be assigned r and the number of control inputs m.

Procedure I for $r \leq m$

Assume that the first row of Θ_{1R} and Θ_{1I} is null such that the first row of $\bar{\Theta}$ looks like $(0,0,x,\ldots,x)$ where 0 and x represent a null row vector and an arbitrary row vector of length m, respectively. Obviously, Θ_{1R} and Θ_{1I} are rank deficient in this case because their first rows are null. Observing the matrix operation $\bar{\Theta}\bar{P} = \bar{Q}$, Eq. (15), one has that $(0,0,x,\ldots,x)(\bar{P}_1^T, \bar{P}_2^T, \bar{P}_3^T, \ldots, \bar{P}_s^T)^T = (x,\ldots,x)(\bar{P}_3^T, \ldots, \bar{P}_s^T)^T = \bar{Q}_1$, where $\bar{Q}_1$ is a row vector denoting some linear combination of the rows of matrix $\bar{Q}$. Thus, the row vector $(x,\ldots,x)$ exists only when the spaces spanned by the row vectors of $(\bar{P}_3^T, \ldots, \bar{P}_s^T)^T$ and $\bar{Q}$ intersect. It is noted, however, that the intersection space generally does exist if complex eigenvalues are assigned along with their respective conjugates. Similarly, let one row of $\bar{\Theta}$ be $(x,x,0,0,\ldots,x)$, then Θ_{2R} and Θ_{2I} are rank deficient since there is a null row vector in these matrices. Such a vector exists only when the spaces spanned by the row vectors of $(\bar{P}_1^T, \bar{P}_2^T, \bar{P}_5^T, \ldots, \bar{P}_s^T)^T$ and $\bar{Q}$ intersect. Note that, in this case, $\bar{P}_3^T$ and $\bar{P}_4^T$ are deleted in contrast to the earlier case where $\bar{P}_1^T$ and $\bar{P}_2^T$ are deleted. The same argument applies to the cases where the 0 vector of length m is in any other position of matrix Θ. At the end, a matrix is generated that has the following form:

$$\hat{\Theta} = \begin{bmatrix} 0 & 0 & x & x & x & x & \cdots & x & x \\ x & x & 0 & 0 & x & x & \cdots & x & x \\ x & x & x & x & 0 & 0 & \cdots & x & x \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x & x & x & x & x & x & \cdots & 0 & 0 \end{bmatrix}$$

where each 0 and x represent, respectively, a null vector and an arbitrary vector of appropriate dimensions. The rows of this matrix can then be used to serve as a basis in solving for the unknown matrix $\bar{\Theta}$ that consists of a number of rank deficient submatrices. For example, it is obvious that any Θ_{1R} and Θ_{1I} generated by the rows of the first and second column matrices in $\hat{\Theta}$ are rank deficient. Such a statement is also true for the other matrices $\Theta_{iR}, \Theta_{iI} (i = 1, \ldots, r)$. Steps to compute the appropriate intersection spaces are shown in the following.

1) Find the dimension of the intersection space of the rows of $\bar{Q}$ and and the matrix $\bar{P}_{1,2}$ defined as

$$\bar{P}_{1,2} = \begin{bmatrix} \bar{P}_3 \\ \bar{P}_4 \\ \vdots \\ \bar{P}_{s-1} \\ \bar{P}_s \end{bmatrix} \tag{16}$$

from the ranks of matrices $\bar{P}_{1,2}$, $\bar{Q}$, and $\begin{bmatrix} \bar{P}_{1,2} \\ \bar{Q} \end{bmatrix}$, i.e.,

$$d_1 = \dim\left[\mathcal{R}_r(\bar{Q}) \cap \mathcal{R}_r(\bar{P}_{1,2})\right]$$

$$= \rho(\bar{P}_{1,2}) + \rho(\bar{Q}) - \rho\left(\begin{bmatrix} \bar{P}_{1,2} \\ \bar{Q} \end{bmatrix}\right) \tag{17}$$

Here d_1 denotes the dimension of the intersection space, $\mathcal{R}_r(\ )$ represents the range of the row space of ( ), and $\rho(\ )$ denotes the rank of ( ). The subscript 1,2 of matrix $\bar{P}_{1,2}$ means that $\bar{P}_{1,2}$ is formed from matrix $\bar{P}$ by deleting submatrices $\bar{P}_1$ and $\bar{P}_2$. Therefore, $\bar{P}_{i,i+1}$ of dimension $m(s-2) \times 2\bar{n}$ is a matrix obtained from $\bar{P}$ by deleting the $\bar{P}_i$ and $\bar{P}_{i+1}$ partition elements. If the dimension d_1 of the intersection space is zero, then all of the s desired closed-loop eigenvalues may not be assignable. On the other hand, if the dimension d_1 is larger than zero, then an orthonormal basis spanning the intersection space may be obtained by taking the QR decompositions of the transpose of matrices $\bar{P}_{1,2}$ and $\bar{Q}$,[15] i.e.,

$$\bar{Q}^T = Q_{\bar{Q}} R_{\bar{Q}}; \qquad \bar{P}_{1,2}^T = Q_{\bar{P}_{1,2}} R_{\bar{P}_{1,2}} \tag{18}$$

where $Q_{\bar{Q}}$ and $Q_{\bar{P}_{1,2}}$ are matrices with orthonormal columns of dimensions $2\bar{n} \times m$ and $2\bar{n} \times (s-2)m$, respectively, $\bar{n}$ is the dimension of the right null space of W, and $R_{\bar{Q}}$ and $R_{\bar{P}_{1,2}}$ are, respectively, $m \times m$ and $(s-2)m \times (s-2)m$ upper triangular matrices. The columns of $Q_{\bar{Q}}$ and $Q_{\bar{P}_{1,2}}$ collect sets of orthonormal basis vectors spanning the range space of rows of $\bar{Q}$ and $\bar{P}_{1,2}$, respectively. Project the basis vectors $Q_{\bar{P}_{1,2}}$ onto $Q_{\bar{Q}}$ to obtain

$$\hat{Q} = Q_{\bar{P}_{1,2}}^T Q_{\bar{Q}} \tag{19}$$

Then, the singular values of matrix $\hat{Q}$ of dimension $m(s-2) \times m$ are the cosines of the principal angles of the subspace pair $\{\mathcal{R}_c(\bar{P}_{1,2}^T), \mathcal{R}_c(\bar{Q}^T)\}$[15] where $\mathcal{R}_c(\ )$ denotes the column space of ( ). Taking the singular value decomposition of $\hat{Q}$ gives

$$\hat{Q} = Y\Sigma Z^T \tag{20}$$

in which Y of dimension $m(s-2)\times m(s-2)$ and Z of dimension $m\times m$ are unitary matrices, and matrix Σ of dimension $m(s-2)\times m$ contains the singular values of $\hat{Q}$, i.e.,

$$D = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & & \sigma_m \\ \vdots & & \vdots \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \equiv \begin{bmatrix} \cos\theta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & & \cos\theta_m \\ \vdots & & \vdots \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \tag{21}$$

The quantities $\sigma_1,\ldots,\sigma_m$ are the singular values, and $\theta_1,\ldots,\theta_m$ are the principal angles of the subspace pair. Define the index j such that

$$1 = \cos\theta_1 = \cdots = \cos\theta_j > \cos\theta_{j+1}$$

then $\mathcal{R}_c(\bar{P}_{1,2}^T) \cap \mathcal{R}_c(\bar{Q}^T)$ is spanned by the range space of the orthonormal columns of H_1^T defined as

$$H_1^T = Q_{P_{1,2}}\bar{Y} \tag{22}$$

where $\bar{Y}$ is the matrix obtained by only keeping the first j columns of matrix Y. Subsequently, an orthonormal basis spanning the intersection space of the rows of $\bar{Q}$ and $\bar{P}_{1,2}$ is represented by the rows of matrix H_1. Note that if the matrix $\bar{P}_{1,2}$ in Eq. (16) does not have a full rank, then only those vectors in $Q_{P_{1,2}}$ that are in the column space of $\bar{P}_{1,2}$ should be used in the equation.

2) Follow the same procedure for matrices $\bar{P}_{i,i+1}$, $i=3,5,\ldots,s-1$, and obtain the dimensions $d_2,\ldots,d_r$ and the corresponding orthonormal bases $H_1,\ldots,H_r$ of their intersection spaces with the range space of the rows of matrix $\bar{Q}$. A collection of m row vectors to be used as basis vectors for computing the rank deficient matrices $\Theta_{1R},\Theta_{1I},\ldots,\Theta_{rR},\Theta_{rI}$ may now be defined if m linearly independent vectors can be chosen from the intersection spaces $H_1,\ldots,H_r$ such that, every one of the intersection spaces contains at least one of the m vectors. These vectors may be chosen through a sequential process wherein first r independent vectors are chosen from $H_1,\ldots,H_r$, each from one intersection space, and the remaining $m-r$ vectors are then arbitrarily picked from the intersection spaces. Denoting these m vectors by $h_1,h_2,\ldots,h_m$ they can be expressed as a linear combination of the rows of matrices $\bar{P}_{i,i+1}$, $i=1,3,\ldots,s-1$, i.e.,

$$\begin{bmatrix} h_1 \\ \vdots \\ h_{k_1} \end{bmatrix} = \hat{\Theta}_1\bar{P}_{1,2} = \hat{\Theta}_1 \begin{bmatrix} \bar{P}_3 \\ \bar{P}_4 \\ \vdots \\ \bar{P}_{s-1} \\ \bar{P}_s \end{bmatrix}$$

$$\vdots \tag{23}$$

$$\begin{bmatrix} h_{k_m} \\ \vdots \\ h_m \end{bmatrix} = \hat{\Theta}_r\bar{P}_{s-1,s} = \hat{\Theta}_r \begin{bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \vdots \\ \bar{P}_{s-3} \\ \bar{P}_{s-2} \end{bmatrix}$$

The quantities $\hat{\Theta}_1,\ldots,\hat{\Theta}_r$ in Eq. (23) are coefficient matrices or vectors, and the indices $k_1,\ldots,k_m$ depend on how the vectors $h_1,h_2,\ldots,h_m$ are defined. Solving Eq. (23) for the matrices or vectors $\hat{\Theta}_1,\ldots,\hat{\Theta}_r$ gives

$$\hat{\Theta}_1 = \begin{bmatrix} h_1 \\ \vdots \\ h_{k_1} \end{bmatrix} \begin{bmatrix} \bar{P}_3 \\ \bar{P}_4 \\ \vdots \\ \bar{P}_{s-1} \\ \bar{P}_s \end{bmatrix}^{\dagger}$$

$$\vdots \tag{24}$$

$$\hat{\Theta}_r = \begin{bmatrix} h_{k_m} \\ \vdots \\ h_m \end{bmatrix} \begin{bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \vdots \\ \bar{P}_{s-3} \\ \bar{P}_{s-2} \end{bmatrix}^{\dagger}$$

where $(\ )^{\dagger}$ denotes a pseudo-inverse. The rows of matrices $\Theta_{1R},\Theta_{1I},\ldots,\Theta_{rR},\Theta_{rI}$ may now be expanded in terms of a collection of basis vectors represented by the rows of matrix $\hat{\Theta}$, i.e.,

$$\bar{\Theta} = D \begin{bmatrix} 0 & \hat{\Theta}_{1,1} & \hat{\Theta}_{1,2} & \cdots & \hat{\Theta}_{1,r-1} \\ \hat{\Theta}_{2,1} & 0 & \hat{\Theta}_{2,2} & \cdots & \hat{\Theta}_{2,r-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \hat{\Theta}_{r,1} & \hat{\Theta}_{r,2} & \cdots & \hat{\Theta}_{r,r-1} & 0 \end{bmatrix} \equiv D\hat{\Theta} \tag{25}$$

Matrix D denotes the coefficient vectors associated with the set of basis vectors and $\hat{\Theta}_{i,k}$ are appropriate partitions of $\hat{\Theta}_i$. It is apparent from Eqs. (25) that since each partition of the matrix $\hat{\Theta}$ contains at least one row of zeros, then each partition matrix is rank deficient and, therefore, may serve as a basis for the rank deficient matrices $\Theta_{1R},\Theta_{1I},\ldots,\Theta_{rR},\Theta_{rI}$. Note that with this approach, the additional freedom beyond eigenvalue assignment is incorporated in the process through which the vectors, $\hat{\Theta}_1,\ldots,\hat{\Theta}_r$, are generated.

Procedure II for $r > m$

When $r > m$, a set of basis vectors for the row space of matrix $\bar{\Theta}$ cannot be obtained following the previous procedure. Instead, a somewhat different approach is followed. First, m intersection spaces $H_1,\ldots,H_m$ are obtained through the intersection of the rows of matrices $\bar{P}_{1,2},\bar{P}_{3,4},\ldots,\bar{P}_{2m-1,2m}$, and matrix $\bar{Q}$, respectively, such that

$$H_1 = \mathcal{R}_r(\bar{P}_{1,2}) \cap \mathcal{R}_r(\bar{Q})$$

$$H_2 = \mathcal{R}_r(\bar{P}_{3,4}) \cap \mathcal{R}_r(\bar{Q})$$

$$\vdots \tag{26}$$

$$H_m = \mathcal{R}_r(\bar{P}_{2m-1,2m}) \cap \mathcal{R}_r(\bar{Q})$$

where matrices $\bar{P}_{1,2}, \bar{P}_{3,4}, \ldots, \bar{P}_{2m-1,2m}$ may be defined as

$$\bar{P}_{1,2} = \begin{bmatrix} \bar{P}_3 \\ \bar{P}_4 \\ \vdots \\ \bar{P}_{2m-1} \\ \bar{P}_{2m} \\ \hat{P}_{2m+1} \\ \hat{P}_{2m+2} \\ \vdots \\ \hat{P}_{s-1} \\ \hat{P}_s \end{bmatrix} \cdots \quad \bar{P}_{2m-1,2m} = \begin{bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \vdots \\ \bar{P}_{2m-3} \\ \bar{P}_{2m-2} \\ \hat{P}_{2m+1} \\ \hat{P}_{2m+2} \\ \vdots \\ \hat{P}_{s-1} \\ \hat{P}_s \end{bmatrix} \tag{27}$$

Note that each matrix $\hat{P}_i$, $i = 2m+1, \ldots, s$ is generated from its corresponding partition matrices $\bar{P}_i$ by replacing one of their rows with a null vector (same rows for complex conjugate pairs). The choice of rows to be replaced as well as the ordering of the partition matrices $\bar{P}_i$, $i = 1, \ldots, s$ is generally arbitrary as long as the resulting intersection space is not null. It should be noted that matrix $\bar{P}_{i,j}$, given in Eq. (27), is not generally the same as the matrix given in Eq. (16). The same notation was mainly used to ease further developments. Now, the steps described in the previous procedure may be followed to obtain a set of m linearly independent vectors $h_1, h_2, \ldots, h_m$ from the intersection spaces $H_1, \ldots, H_m$. Expanding the vectors $h_1, h_2, \ldots, h_m$, one has

$$\begin{aligned} h_1 &= \hat{\Theta}_1 \bar{P}_{1,2} \\ &\vdots \\ h_m &= \hat{\Theta}_m \bar{P}_{2m-1,2m} \end{aligned} \tag{28}$$

Solving for the vectors $\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_m$ gives

$$\begin{aligned} \hat{\Theta}_1 &= h_1 \bar{P}_{1,2}^{\dagger} \\ &\vdots \\ \hat{\Theta}_m &= h_m \bar{P}_{2m-1,2m}^{\dagger} \end{aligned} \tag{29}$$

The rows of the rank deficient matrices $\Theta_{1R}, \Theta_{1I}, \ldots, \Theta_{rR}, \Theta_{rI}$ may now be expanded in terms of a collection of basis represented by the rows of matrix $\hat{\Theta}$, i.e.,

$$\bar{\Theta} = D \begin{bmatrix} 0 & \hat{\Theta}_{1,1} & \hat{\Theta}_{1,2} & \cdots & \hat{\Theta}_{1,m} & \hat{\Theta}_{1,m+1} & \cdots & \hat{\Theta}_{1,r-1} \\ \hat{\Theta}_{2,1} & 0 & \hat{\Theta}_{2,2} & \cdots & \hat{\Theta}_{2,m} & \hat{\Theta}_{2,m+1} & \cdots & \hat{\Theta}_{2,r-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hat{\Theta}_{m,1} & \hat{\Theta}_{m,2} & \cdots & 0 & \hat{\Theta}_{m,m} & \cdots & \cdots & \hat{\Theta}_{m,r-1} \end{bmatrix} \equiv D\hat{\Theta} \tag{30}$$

Matrix D denotes the coefficient vectors associated with the set of basis vectors and $\hat{\Theta}_{i,k}$ are appropriate partitions of $\hat{\Theta}_i$. It is apparent from Eq. (30) that since each partition of the matrix $\hat{\Theta}$ contains at least one row or column of zeros, then each partition matrix is rank deficient and, therefore, may serve as a basis for the rank deficient matrices Θ_{1R}, $\Theta_{1I}, \ldots, \Theta_{rR}, \Theta_{rI}$. Again, with this approach, the additional freedom beyond eigenvalue assignment is incorporated in the process through which the vectors, $\hat{\Theta}_1, \ldots, \hat{\Theta}_m$, are generated.

Equation (25) or (30) is substituted into Eq. (15) to yield

$$\bar{\Theta}\bar{P} = D\hat{\Theta}\bar{P} = \bar{Q} \tag{31}$$

The coefficient matrix D, as well as matrices $\Theta_{1R}, \Theta_{1I}, \ldots$, Θ_{rR}, Θ_{rI}, may now be computed from Eq. (31):

$$D = \bar{Q}[\hat{\Theta}\bar{P}]^{\dagger} \tag{32}$$

and

$$\bar{\Theta} = \bar{Q}[\hat{\Theta}\bar{P}]^{\dagger}\hat{\Theta} \tag{33}$$

Finally, having computed the coefficient $\Theta_{1R}, \Theta_{1I}, \ldots, \Theta_{rR}, \Theta_{rI}$ from Eq. (33), the gain matrix G is determined using Eq. (13), yielding

$$G = \left[(\Theta_1 + \hat{V}_1) \quad \ldots \quad (\Theta_s + \hat{V}_s)\right] W^{\dagger} \tag{34}$$

Concluding Remarks

A new approach for the eigenvalue assignment via output feedback for linear, first-order, time-invariant systems has been presented. The approach uses subspace intersection techniques to assign the closed-loop eigenvalues to desired locations. It can assign the maximum allowable number of closed-loop eigenvalues and is computationally stable since singular value decomposition (SVD) and/or QR decomposition are used throughout the computational procedure. However, similar to other eigensystem assignment techniques with output feedback, the stability of the resulting closed-loop system is not guaranteed. Perhaps, the freedom identified beyond the eigensystem assignment may be exploited to ensure closed-loop stability.

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Decomposition Method for Solving Weakly Coupled Algebraic Riccati Equation

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I. Introduction

THE linear weakly coupled systems have been studied in different setups by many researchers.[1-15] The main equation of the linear optimal control theory—the Riccati equation—can be obtained from the Hamiltonian matrix. For weakly coupled systems, the Hamiltonian matrix retains the weakly coupled form by interchanging some of the state and costate variables so that it can be block diagonalized via the decoupling transformation introduced in Ref. 13. The main idea of this paper is to obtain the solution of the global algebraic Riccati equation from two decoupled reduced-order subsystems, both leading to the nonsymmetric algebraic Riccati equations that can be solved simultaneously. It has been shown that such a solution exists under stabilizability-detectability conditions imposed on both subsystems.

II. Exact Reduced-Order Algebraic Riccati Equations

Consider the linear weakly coupled system

$$\dot{x}_1 = A_1x_1 + \epsilon A_2x_2 + B_1u_1 + \epsilon B_2u_2, \quad x_1(t_0) = x_{10}$$

$$\dot{x}_2 = \epsilon A_3x_1 + A_4x_2 + \epsilon B_3u_1 + B_4u_2, \quad x_2(t_0) = x_{20} \tag{1}$$

with

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = D\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} D_1 & \epsilon D_2 \\ \epsilon D_3 & D_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2}$$

where $x_i \epsilon R^{n_i}$, $u_i \epsilon R^{m_i}$, $z_i \epsilon R^{r_i}$, $i = 1, 2$, are state, control, and output variables, respectively. The system matrices are of appropriate dimensions and, in general, they are bounded functions of a small coupling parameter ϵ.[10-12] In this paper we will assume that all given matrices are constant.

With Eqs. (1) and (2), consider the performance criterion

$$J = \frac{1}{2}\int_{t_0}^{\infty}\left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T D^TD\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T R\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right\} dt \tag{3}$$

with positive definite R, which has to be minimized. It is assumed that matrix R has the weakly coupled structure, that is,

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \tag{4}$$

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The optimal closed-loop control law has the very well-known form[16]

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -R^{-1}\begin{bmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{bmatrix}^T P\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -R^{-1}B^TPx \tag{5}$$

where P satisfies the algebraic Riccati equation given by

$$0 = PA + A^TP + Q - PSP \tag{6}$$

with

$$A = \begin{bmatrix} A_1 & \epsilon A_2 \\ \epsilon A_3 & A_4 \end{bmatrix}, \quad S = BR^{-1}B^T = \begin{bmatrix} S_1 & \epsilon S_2 \\ \epsilon S_2^T & S_3 \end{bmatrix} \tag{7}$$

and

$$Q = D^TD = \begin{bmatrix} Q_1 & \epsilon Q_2 \\ \epsilon Q_2^T & Q_3 \end{bmatrix} \tag{8}$$

The open-loop optimal control problem of Eqs. (1-4) has the solution given by

$$u(t) = -R^{-1}B^Tp(t) \tag{9}$$

where $p(t)\epsilon R^{n_1+n_2}$ is a costate variable satisfying[16]

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{10}$$

Partitioning p into $p_1 \epsilon R^{n_1}$ and $p_2 \epsilon R^{n_2}$ such that $p = [p_1^T \; p_2^T]$, and rearranging rows in Eq. (10), we can get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{p}_1 \\ \dot{x}_2 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} T_1 & \epsilon T_2 \\ \epsilon T_3 & T_4 \end{bmatrix}\begin{bmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix} \tag{11}$$

where T_i's, $i = 1, 2, 3, 4$, are given by

$$T_1 = \begin{bmatrix} A_1 & -S_1 \\ -Q_1 & -A_1^T \end{bmatrix}, \quad T_2 = \begin{bmatrix} A_2 & -S_2 \\ -Q_2 & -A_3^T \end{bmatrix}$$

$$T_3 = \begin{bmatrix} A_3 & -S_2^T \\ -Q_2^T & -A_2^T \end{bmatrix}, \quad T_4 = \begin{bmatrix} A_4 & -S_3 \\ -Q_3 & -A_4^T \end{bmatrix} \tag{12}$$

Introducing a notation

$$\begin{bmatrix} x_1 \\ p_1 \end{bmatrix} = w, \qquad \begin{bmatrix} x_2 \\ p_2 \end{bmatrix} = \lambda \tag{13}$$

and applying the transformation introduced in Ref. 13

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = K^{-1}\begin{bmatrix} w \\ \lambda \end{bmatrix} \tag{14}$$

$$K = \begin{bmatrix} I & -\epsilon L \\ \epsilon H & I - \epsilon^2 HL \end{bmatrix}, \quad K^{-1} = \begin{bmatrix} I - \epsilon^2 LH & \epsilon L \\ -\epsilon H & I \end{bmatrix} \tag{15}$$

where L and H satisfy

$$T_1L + T_2 - LT_4 - \epsilon^2 LT_3L = 0 \tag{16}$$

$$H(T_1 - \epsilon^2 LT_3) - (T_4 + \epsilon^2 T_3L)H + T_3 = 0 \tag{17}$$

will produce a decoupled form

$$\dot{\eta} = (T_1 - \epsilon^2 LT_3)\eta \tag{18}$$

$$\dot{\xi} = (T_4 + \epsilon^2 T_3L)\xi \tag{19}$$